Mathematical Model for Handling Unstable Time Series by Using a Linear Approximation Technique

Kawther Abood Neamah
Assistant Professor, Baghdad College of Economics Sciences University, Iraq.
E-mail: cs@baghdadcollege.edu.iq

Received September 14, 2021; Accepted December 14, 2021
ISSN: 1735-188X
DOI: 10.14704/WEB/V19I1/WEB19189

Abstract

Time series are typically built on basic assumptions that include stationarity, linearity and normality. The three characteristics are crucial for estimating and building time series models. Studies on time series include these assumptions. To deal with unstable time series that are based on its basis, mathematical models that are suitable for such series are adopted in this study. A nonlinear self-regression model, called the rational model, is proposed. This model is a fraction in which the numerator is the complete sine function and the denominator is an exponential self-regression model. The fixed point and limit cycle of the model are simulated and determined, and its stability is studied using a linear approximation technique.

Keywords

Limit Cycle, Stability, Single Point, Rational Model, Nonlinear Autoregressive Model.

Introduction

A time series is one of the most important and useful mathematical methods for research and scientific studies. It is considered the pillar of development plans and the primary response to some problems and changes in the medical, economic and service aspects. The study of time series includes basic assumptions, such as being linear, gradual and natural. It also involves determining how unstable time series should be treated to build appropriate mathematical models. For example, whether linear or nonlinear models should be adopted, such that the parameter of the model is a rational function simplified by the cosine function and the denominator is an exponential autoregression.

This model exhibits a nonlinear behaviour in the form of limit cycles, and the linear approximation method proposed by Ozaki (1985) was used to determine the stability of this model and other statistical properties.

Basic Concepts

1. Autoregressive Model

The discrete-type autoregressive time series model is one of the most widely used models in different applications. The general form of the p-rank autoregressive model, which is symbolised by AR(P), is

$$X_t = M + a_1 X_{t-1} + a_2 X_{t-2} + \cdots + a_P X_{t-P} + Z_t, \quad (1\text{-}1)$$

where $a_1, a_2, \ldots, a_P, M$ are constants; and $\{Z_t\}$ is a purely random, noncorrelated process called white disturbance. It exhibits the following characteristics:

$cov(Z_t, Z_{t+k}) = o, \forall k \neq o;$
$Var(Z_t) = \sigma_z^2;$
$E(Z_t) = o.$

The model, Equation (1-1), can be written in the following form:

$$\propto (B) X_t = M + Z_t \qquad (1\text{-}2)$$

Given that $\propto (B) = 1 - a_1 B - a_2 B^2 - \cdots - a_P B^P$, $and\ B$ represents the background displacement effect, which is known as $B^n X_t = X_{t-n}$, all $n = 0,1,2,3, \ldots,$ and it typically adopts the autoregressive model as follows:

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \cdots + a_P X_{t-P} + Z_{t.} \qquad (1\text{-}3)$$

The general solution for Equation (1-3) is

$$X_t = f(t) + \propto^{-1} (B) Z_t. \qquad (1\text{-}4)$$

$f(t)$ is the complement function that represents the solution for the homogeneous differential equation $\propto (B) X_t = o$, which exhibits the following form:

$$f(t) = A_1 \lambda_1^t + A_2 \lambda_2^t + \cdots A_P \lambda_P^t \qquad (1\text{-}5)$$

where $A_1, A_2, \ldots, A_P$ are optional constants; and $\lambda_1, \lambda_2, \ldots \lambda_P$ are the roots of the characteristics equation.

$$\lambda^P - \sum_{j=1}^{P-1} a_i \lambda^{P-1} = o \qquad (1\text{-}6)$$

Wan $\propto^{-1} (B) Z_t$ represents the model's own solution.

The autoregressive model, AR(P), is apparently stable if and only if $|\lambda_i| < 1$ for each $i = 1,2,3, \dots, P$ [7].

2. Relative Autoregressive Model

Let $\{X_t\}$ be a time series, where $t = o, \mp 1, \mp 2, \dots$. The relative model of the autoregressive rank P is defined as the model that fulfils the following relationship:

$$X_t = \frac{P_1}{P_2} X_{t-i} + Z_t; \quad i = 1,2, \dots, P.$$

Given that P_1 and P_2 are polynomials or linear or nonlinear functions,
we assume that

$$P_1(X) = \sum_{i=1}^{P} \propto cos\, cos\, (2\pi w_i t + \emptyset)\,,$$

$$P_2(X) = \sum_{i=1}^{P} (\pi_i + \theta_i e^{-Ye_{i-1}^2}).$$

If $P_1(.)$ represents the cosine function; $\emptyset, w, \propto$ are constants; $P_2(\ )$ represents an Asian model and π, Y are constants, then $\{Z_t\}$ is the model series as follows:

$$X_t = \sum_{i=1}^{P} \frac{\propto cos\, (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-Ye_{i-1}^2}} X_{i-1} + Z_t \qquad (1\text{-}7)$$

3. Fixed point [5][6]

Consider the following model:

$$X_t = f(X_{t-1}, X_{t-2}, \dots, X_{t-P}). \qquad (1\text{-}8)$$

The single point $\in$ in the model, i.e. Equation (1-8), is defined as the point that the model's path is approaching. If the path approaches $\in$ when $t \to \infty$, then it is a stable point.

However, if the path approaches $\in$ when $t \to -\infty$, then $\in$ is a single unstable point. The necessary and sufficient condition for $\in$ is that the following relationship must be fulfilled:

$$\in = f(\in, \in, \in, \dots, \in).$$

4. Limit Cycle

Consider the following model $X_t = f(X_{t-1}, X_{t-2}, \dots, X_{t-P})$, which defines the terminal period of the above model as the isolated and closed path $X_t = X_t, X_{t+1}, X_{t+2}, \dots, X_{t+q}$ because q represents a positive integer.

An isolated path is defined as any path that starts extremely close to a limit cycle and approaches it when $t \to \infty$ or $t \to -\infty$.

If the path approaches it when $t \to \infty$, then it is called a stable limit cycle. If the path approaches it when $t \to \infty$, then it is called an unstable limit cycle.

For the closed path, if the initial values $X_1, X_2, X_3, \dots, X_P$ belong to a limit cycle, then $(X_1, X_2, \dots, X_P) = (X_{1+kq}, \dots, X_{P+kq})$ for every positive integer k, where P is the period for the limit cycle [6].

5. Linear Approximation Method [5]

The local linear approximation method was proposed by Ozak (1985) for finding the stability of nonlinear models. This method is summarized as follows.

- First stage: Finding the single nonzero point of the nonlinear model.
- Second stage: Testing the stability of this point by using a linear approximation technique.

Stability of the Relative Autoregressive Model

3.1 Stability [2]

In many applications in engineering physics, we encounter operations that can be described as statistical equilibrium. That is, if we make observations of this type of process and divide them into groups of periods, then the different sections of these observations will appear similar. More accurately, the statistical qualities remain constant and do not change with time.

Stochastic processes that behave in this manner are called stationary processes. That is, no growth or decay of the time series data occurs; data are scattered around a constant medium and have a constant variance. Therefore, $X_1, X_2, X_3, \dots, X_t$ must have the same probability density function, i.e.

$$f(X_1, X_2, X_3, \dots, X_t) = f(X_{1+k}, X_{2+k}, \dots X_{t+k}).$$

k represents a real number,
and the joint probability distribution does not change with a change in time or when shifting with fixed numbers.

2. Stability of Linear Autoregressive Models

The complement function $f(t)$ in Equation (1-5), which represents the solution of the homogeneous differential equation $\propto (B)X_t = o$, approaches zero when $t \to \infty$ if and only if the absolute value of all the roots of the characteristic equation are less than one. The characteristic is located inside the unit circle, which is a circle with the centre as the origin and a radius of 1.

3. Stability of Nonlinear Models

The techniques developed to study the stability of linear time series models are highly dependent on the default being linear, and thus, they cannot be easily expanded to the nonlinear state. We study special cases of nonlinear models. Many studies on the stability of nonlinear models have been conducted. They include finding stability in accordance with Lyapunov's direct theory, which depends on the stability of the initial model and on more than one variable.

The stability of the Lagrange method, which is typically limited to one or several chains and must have a definite string, i.e. $m \geq |X_i|$, because M is a constant, it is used to find the stability of continuous time series or intermittent time. We adopt the linear approximation method proposed by Ozaki (1985) to study the stability of nonlinear models, focusing on one of the relative autoregressive models.

4. Stability of the Autoregressive Model

Finding the stability of relativistic models is a difficult task, and thus, we adopt special cases and simplified formulas.

Stability can be found using the formula for stability of linear models. For example, the stability of can be tested the following proportional model and its treatment as linear models:

$$Y(t) = \frac{0.99 + e^2(t-1)}{1 + 0.1y^2(t-1) + e^2(t-1)} y(t-1) + z(t). \qquad (2\text{-}1)$$

This model can be expressed as follows:

$$Y(t) = \varphi y(t-1) + z(t).$$

Given that

$$\varphi = \frac{0.99 + e^2(t-1)}{1 + 0.1\, y^2(t-1) + e^2(t-1),}$$

the characteristic equation for this model is $V - \varphi = o$.

Thus, the condition of stability $|M| < 1$ because M is the root of the characteristic equation, given that

$$o < \frac{0.99 + e^2(t-1)}{1 + 0.1\, y^2(t-1) + e^2(t-1)} < 1.$$

The model is stable within the period of $-1 < m < 1$. A number of researchers have studied the stability of these linear and nonlinear models in special cases. He (?) investigated exponential autoregressive models by using the linear approximation method of Ozaki (1985). Ozaki researched nonlinear random oscillations in kinetic systems and modelled them by using the exponential autoregressive model (xxxx).

In the succeeding sections, we study the stability of the relative autoregressive model in a special case wherein $P_1(\,.\,)$ trigonometric functions, $P_2(\,.\,)$. An Asian model, which is the proposed model. Studying the stability of this model involves finding a single point. We test the stability of this model by using the linear approximation method or the limit cycle.

Let's begin with the cases for Model (1-7) and generalise the idea to include a model with rank P.

5. Finding the Single Point of the Proposed Model

Model (1-7):

$$X_t = \sum_{i=1}^{P} \frac{\propto cos\,(2\pi w_i t + \emptyset)}{\pi_i + \emptyset_i e^{-YX_{t-1}^2}} X_{t-i} + Z_{t.}$$

$\propto, \emptyset, w, \pi, \emptyset$ $and\ y$ are predefined; and $\{Z_t\}$ noises when $P = 1$, we obtain

$$X_t = \frac{\propto cos\,(2\pi w_1 t + \emptyset)}{\pi_i + \emptyset_1 e^{-YX_{t-1}^2}} X_{t-i} + Z_{t,}\ \text{(2-2)}$$

which is a first-order relative model. We assume that the effect of $\{Z_t\}$ is equal to zero, and then we use the single point definition that we obtain.

$$\in = \frac{\propto coscos\,(2\pi w_1 t + \emptyset)}{\pi_i + \emptyset_1 e^{-YX^2}} \in \qquad (2\text{-}3)$$

$\in = o$ represents the zero fixed point, which is one of solutions called the trivial solution, and the rest.

The fixed (nonzero) points can be determined by solving Equation (2-3) as follows:

$$1 = \frac{\propto cos\,(2\pi w_1 t + \emptyset)}{\pi_1 + \emptyset_1 e^{-Ye^2}}$$
$$\Longrightarrow \emptyset_1 e^{-Y\in^2} = \propto cos\,cos\,(2\pi w_1 \emptyset) - \pi_1$$
$$\Longrightarrow -Y \in^2 = Ln[(\propto cos\,cos\,(2\pi w_1 t + \emptyset) - \pi_1)/\theta_1]$$
$$\Longrightarrow \in^2 = \frac{-Ln(\propto cos\,cos\,(2\pi w_1 \emptyset) - \pi_1)}{\frac{\emptyset_1}{y}}$$
$$\Longrightarrow \in = \pm\sqrt{\frac{-Ln\left[\frac{\propto coscos\,(2\pi w_1 t + \emptyset) - \pi}{\theta_1}\right]}{y}}. \qquad (2\text{-}4)$$

If it exists, this solution represents two single corresponding points, i.e. if

$$\in_1 = \sqrt{\frac{-Ln\left[\frac{\propto coscos\,(2\pi w_1 t + \emptyset) - \pi_1}{\theta_1}\right]}{y}},$$

$$\in_2 = \sqrt{\frac{-Ln\left[\frac{\propto coscos\,(2\pi w_1 t + \emptyset) - \pi_1}{\theta_1}\right]}{y}}.$$

The single point of the relativistic model of order P.

To find the single point of the relativistic model with a P rank, we disregard the effect of $\{Z_t\}$ on the proposed model, i.e. Equation (1-7), and thus, we obtain

$$X_t = \sum_{i=1}^{P} \frac{\propto cos\,(2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-YX_{t-1}^2}} X_{t-i}.$$

By using the single point definition, we derive

$$\in = \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-YX_{t-1}^2}} \in,$$

where $\in = o$ is one of the solutions and called the zero point of the model, Equation (1-7).

To find the remaining single points for any given rank, a formula similar to the method used for finding individual points when $P = 1$ is adopted.

6. Necessary Conditions for the Model to have a Limit Cycle

Consider the following model: $X_t = f(X_{t-1}, X_{t-2}, \dots, X_{t-P})$, where $f(.)$ is an exponential function.

The necessary conditions that must be satisfied for the model to have a limit cycle are as follows.

1. All roots of the characteristic equation should lie inside the unit circle when X_{t-1} is extremely large, i.e. $(|X_{t-1}| \to \infty)$.
2. At least one of the roots of the characteristic equation should lie inside the unit circle when X_{t-1} is extremely small, i.e. $(|X_{t-1}| \to \infty)$.
3. $\frac{1-\sum_{i=1}^{P} \varphi_i}{\sum_{i=1}^{P} \pi_i} < 1 \quad or \quad \frac{1-\sum_{i=1}^{P} \varphi_i}{\sum_{i=1}^{P} \pi_i} > o.$

7. Limit Cycle for the Proposed Model

We find a limit cycle by using Equation (6-3) and after disregarding the effect of disturbances, as follows.

1. When X_{t-1} is extremely large, i.e. when it is $|X_{t-1}| \to \infty$, then we obtain magnitude $e^{-X_{t-1}^2} \to o$

By substituting it into Equation (7-1), we derive

$$X_t = \sum_{i=1}^{P} \frac{\propto cos\ (2\pi w_i t + \emptyset)}{\pi_i} X_{i-1},$$

which can be written as follows:

$$X_t = A_i X_{i-1}; i = 1,2,3, \dots, P; \qquad (2\text{-}5)$$

because

$$A_i = \sum_{i=1}^{P} \frac{\propto cos\ (2\pi w_i t + \emptyset)}{\pi_i}$$

and its distinctive equation

$$V^P - A_1\, V^{P-1} - A_2\, V^{P-2} - \cdots A_P = 0.$$

We obtain the roots by solving this equation.

2. When X_{t-1} is extremely small, i.e. when it is $|X_{t-1}| \to \infty$. Then, we obtain

$$e^{-X_{t-1}^2} \to o.$$

By substituting it into Equation (7-1), we derive

$$X_t = \sum_{i=1}^{P} \frac{\propto cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i} X_{i-1},$$

which is an autoregressive model of rank [P], and its

$$V^P - A_1'\, V^{P-1} - A_2'\, V^{P-2} - \cdots - A_P' = o.$$

Given that

$$A_1' = \sum_{i=1}^{P} \frac{\propto cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i},$$

We obtain the roots by solving the equation.

3. A single nonzero point is present and real if the following condition is met:

$$< 1 \frac{\left[\frac{\propto cos cos\ (2\pi w_1 t + \varphi)\pi_1}{\theta_1}\right]}{Y} < 0.$$

8. Single Point Nonzero Stability of the Proposed Model

We test single point nonzero stability by using a local linear approximation method near the single point as follows.

Given the formulated model, Equation (7-1).

$$X_t = \sum_{i=1}^{P} \frac{\propto cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-YX_{t-1}^2}} X_{t-i} + Z_t$$

We assume that the effect of Z_t is zero and that $i = 0,1,2, \dots, P$; $X_{t-i} = \in + \in_{t-i}$, where $\in$ is an extremely small amount. We also assume $y = 1$.

$$\in + \in_t = \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-(\in + \in_{t-i})^2}} (\in + \in_{t-i}),$$

$$\in + \in_{t-i} = \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta e^{-\in^2 - 2\in \in_{t-1}} - \in_{t-1}^2},$$

where $\in_t^n \to o$ for each $n \geq 2$.

$$\in + \in_t = \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-\in^2 - 2\in \in_{t-1}}} (\in + \in_{t-i})$$

Given that $\in, \in_{t-1}$ is extremely small,

we assume that $2 \in \in_{t-1} = o$, and we obtain

$$\in + \in_t = \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta e^{-\in^2}} (\in + \in_{t-i}),$$

$$\in + \in_t = \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-\in^2}} \in + \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-\in^2}} \in_{t-i},$$

$$\in_t = \left(\sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-\in^2}} - 1 \right) + \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-\in^2}} \in_{t-i}.$$

We substitute the value $\in$ defined in Equation (4-2) as follows:

$$\theta_1 e^{-\in^2} = \theta_i e^{\frac{Ln\left(\frac{\propto coscos\ (2\pi w_i t + \varphi)}{\theta_i}\right)}{Y}}$$

$$= \theta_i e^{\frac{1}{Y} Ln\left[\frac{\propto coscos\ (2\pi w_i t + \varphi)}{\theta_i}\right]}$$

$$= \theta_i e^{Ln[\frac{\propto coscos\ (2\pi w_i t + \varphi)}{\theta_i}]^{\frac{1}{Y}}}$$

$$= \theta_i [\frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\theta_i}]^{\frac{1}{Y}}.$$

The form is transformed into

$$\in_t = \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i [\frac{\propto coscos\ (2\pi w_i t + \emptyset)}{\theta_i}]^{\frac{1}{Y}}} \in_{t-i} +$$

$$\left(\sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i [\frac{\propto coscos\ (2\pi w_i t + \emptyset)}{\theta_i}]^{\frac{1}{Y}}} - 1 \right) \left(\mp \sqrt{\frac{-Ln[\propto cos\ cos\ (2\pi w_i t + \emptyset)\ - \pi_1]}{Y}} \right),$$

$$\in_t = M + \sum_{i=1}^{P} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i [\frac{\propto coscos\ (2\pi w_i t + \emptyset)}{\theta_i}]^{\frac{1}{Y}}} \in_{t-i}.$$

M represents a fixed quantity. The preceding equation represents a linear autoregressive model of rank P without white disturbance, i.e.

$$\in_t = A + h_1 \in_{t-1} + h_2 \in_{t-2} + \cdots + h_P \in_{t-P.} \qquad (2\text{-}6)$$

Given that

$$h_1 = \frac{\propto coscos\ (2\pi w_i t + \emptyset)}{\pi_i + \theta_i [\frac{\propto coscos\ (2\pi w_i t + \emptyset)}{\theta_1}]^{\frac{1}{Y}}}, \qquad (2\text{-}7)$$

$$h_j = \frac{\propto coscos\ (2\pi w_i t + \emptyset)}{\pi_j + \theta_j [\frac{\propto coscos\ (2\pi w_i t + \emptyset)}{\theta_j}]^{\frac{1}{Y}}}; j = 2,3, \dots, P. \qquad (2\text{-}8)$$

By using the stability condition of a linear autoregressive model, we determine that the model is stable if and only if all the roots of the characteristic equation lie within the unit circle.

$$\lambda^P - \sum_{j=1}^{P} hi\lambda^{P-j} + A = o$$

9. Condition of Stability Limit Cycle

If a limit cycle is found in the period [q] of the proposed model, Equation (7-1), the first-order formula of the relative model is

$$X_t = \sum_{i=1}^{P} \frac{\propto cos\,(2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-YX_{t-1}^2}} X_{t-i} + Z_t.$$

When $P = 1$, the following theorem provides the condition of stability in terms of the parameters and constants of the preceding model.

10. 3 Theory

Limit cycle in period q (if any) for the preceding model when $P = 1$ is orbitally stable if the following condition is met:

$$\left| \prod_{i=1}^{q} \left[\frac{\propto cos\, cos\,(2\pi w_i t + \emptyset)}{\pi_i + \theta_i e^{-YX_{t-1}^2}} \right] \right| < 1.$$

Proof:

We assume that the proposed model, Equation (7-1), has a limit cycle with periods q and $q > 1$ in the following form:

$$X_t, X_{t+1}, X_{t+2}, \dots, X_{t+q} = X_t.$$

The path is a closed and isolated. Each X_s point on a path that is close to a limit cycle can be expressed as $X_s + \in_s$, such that

$|\in_s|$ is too small, i.e. $|\in_s|^n \to o$ per $n \geq 2$ for $s = t, t-1$.

By concealing the white annoyance effect of Z_t, replacing $X_t + \in_t$ instead of X_t and $X_{t-1} + \in_{t-1}$ instead of X_{t-1} and assuming $Y = 1$ in Equation (7-1) because $P = 1$, we obtain

$$X_t + \in_t = \frac{\propto cos\, cos\,(2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-(X_{t-1} + \in_{t-1})^2}} (X_{t-1} + \in_{t-1}),$$

$$X_t + \in_t = \frac{\propto cos\, cos\,(2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2 - 2X_{t-1}\in_{t-1} - \in_{t-1}^2}} (X_{t-1} + \in_{t-1}),$$

$$X_t + \in_t = \frac{\propto cos\, cos\,(2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2 - 2X_{t-1}\in_{t-1}}} (X_{t-1} + \in_{t-1}),$$

$$\in_t = \frac{\propto cos\, cos\,(2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2 - 2X_{t-1}\in_{t-1}}} X_{t-1} + \frac{\propto cos\, cos\,(2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2 - 2X_{t-1}\in_{t-1}}} \in_{t-1} - X_t.$$

Given that

$$X_t = \frac{\propto cos\ cos\ (2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2}} X_{t-1}.$$

Then,

$$\in_t = \frac{\propto cos\ cos\ (2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2 - 2X_{t-1}\in_{t-1}}} \in_{t-1}.$$

Considering that $\in_{t-1}$ is extremely small, and $X_t = X_{t-1} = X_{t-2} = \cdots = \in$, then $X_{t-1} \in_{t-1} \cong o$

$$\Longrightarrow \in_t = \frac{\propto coscos\ (2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2}} \in_{t-1} \qquad (2\text{-}9)$$

$$\frac{\in_t}{\in_{t-1}} = \frac{\propto cos\ cos\ (2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2}}.$$

In

$$T(X_{t-1}) = \frac{\propto cos\ cos\ (2\pi w_1 t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t-1}^2}},$$

Equation (9-2) is expressed in the following form:

$\in_t = T(X_{t-1}) \in_{t-1},$
i.e.
$\in_{t+1} = T(X_t) \in_t,$
$\in_{t+2} = T(X_{t+1}) \in_{t+1},$
$\in_{t+3} = T(X_{t+2}) \in_{t+2},$
$\in_{t+q} = T\left(X_{t+q-1}\right) \in_{t+q-1},$

such that

$$\in_{t+q} = T\left(X_{t+q-1}\right).T\left(X_{t+q-2}\right) T\left(X_{t+q-3}\right) \dots T(X_t) \in_t.$$

Therefore,

$$\in_{t+q} = \left| \prod_{i=1}^{q} T\left(X_{t+q-i}\right) \right| \in_t,$$

i.e.

$$\left|\frac{\epsilon_{t+q}}{\epsilon_t}\right| = \left|\prod_{i=1}^{q} T\left(X_{t+q-i}\right)\right|.$$

This ratio must be less than one to converge to zero, i.e.

$$\left|\prod_{i=1}^{q} T\left(X_{t+q-i}\right)\right| < 1$$

or

$$\left|\prod_{i=1}^{q} \frac{\propto cos\ cos\ (2\pi w_i t + \emptyset)}{\pi_1 + \theta_1 e^{-X_{t+q-1}^2}}\right| < 1.$$

That is, the limit cycle is orbital and with this

Example of Model Application

In this section, we review the numerical examples, determine the conditions for their stability and present some forms that exhibit the stability of the proposed model.

1. Example

Given the following model:

$$X_t = \frac{-0.4 * cos\ (7.29 * 3 + 1)}{-0.4 + 0.4e^{-0.4X_{t-1}^2}} X_{t-1} + Z_t.$$

From paragraph (7-3), we determine that the stability condition for this model is

$$0 < \frac{\left[\frac{\propto coscos\ (2\pi w_1 t + \varphi) - \pi_1}{\theta_1}\right]}{Y} < 1.$$

The condition for stability is 0.140956 because the model is stable at $0 < 0.140956 < 1$.

- The following figures illustrate the stability of the model by assuming different initial values.

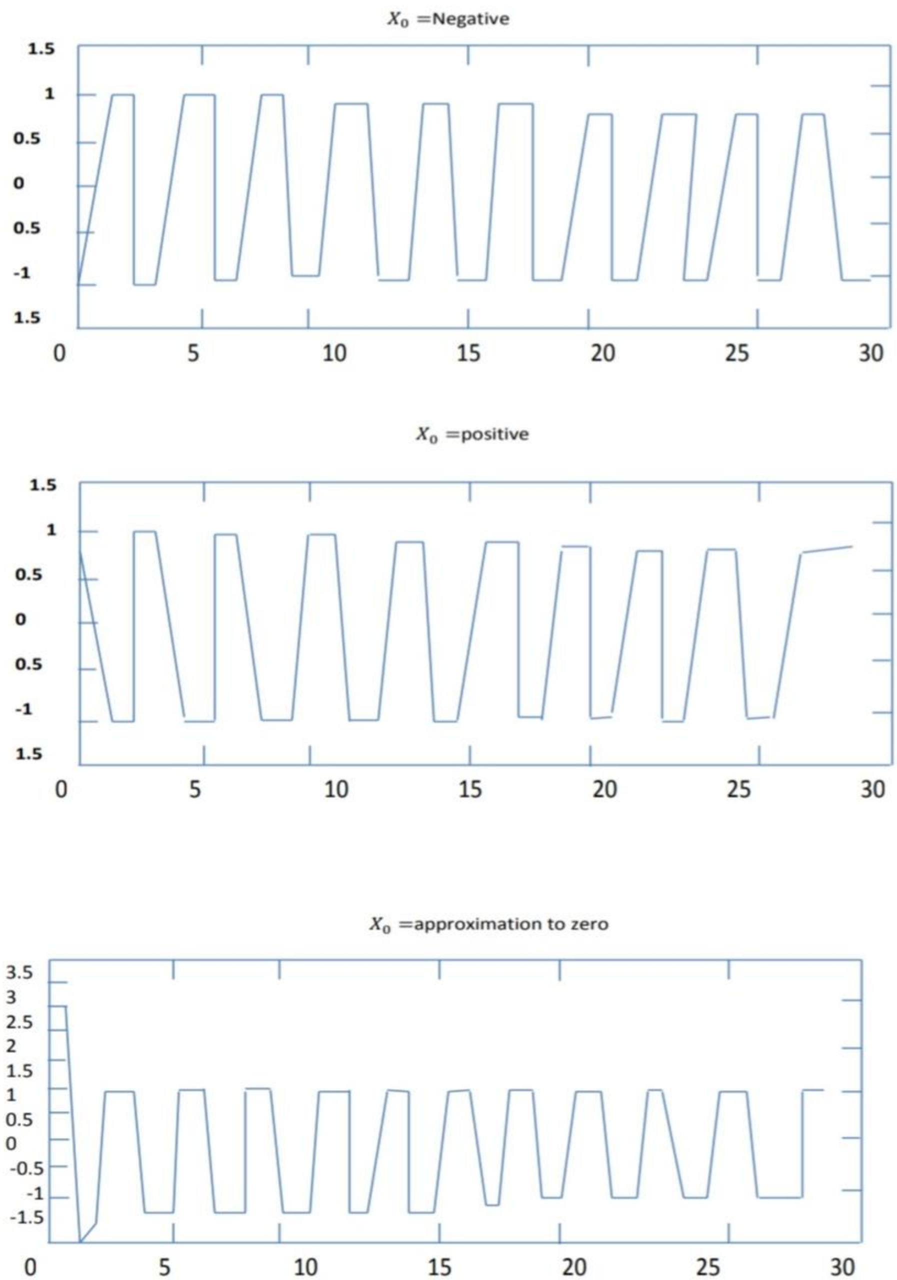

Figure 1

As shown in Figure (1), the generated chain of the model does not depend on the initial condition, and the paths approach a limit cycle.

2. Example

Given the following model:

$$X_t = \frac{-0.4 * cos\ (7.29 * 3 + 0.3)}{-0.4 + 0.4e^{-0.4X_{t-1}^2}} X_{t-1} + Z_t.$$

From paragraph (7-3), we determine that the stability condition for this model is

$$0 < \frac{\left[\frac{\propto coscos\,(2\pi w_1 t+\varphi)-\pi_1}{\theta_1}\right]}{Y} < 1.$$

The condition for stability is 4.380573 because the model is unstable at 4.380573

Figure (2) shows the instability of the model.

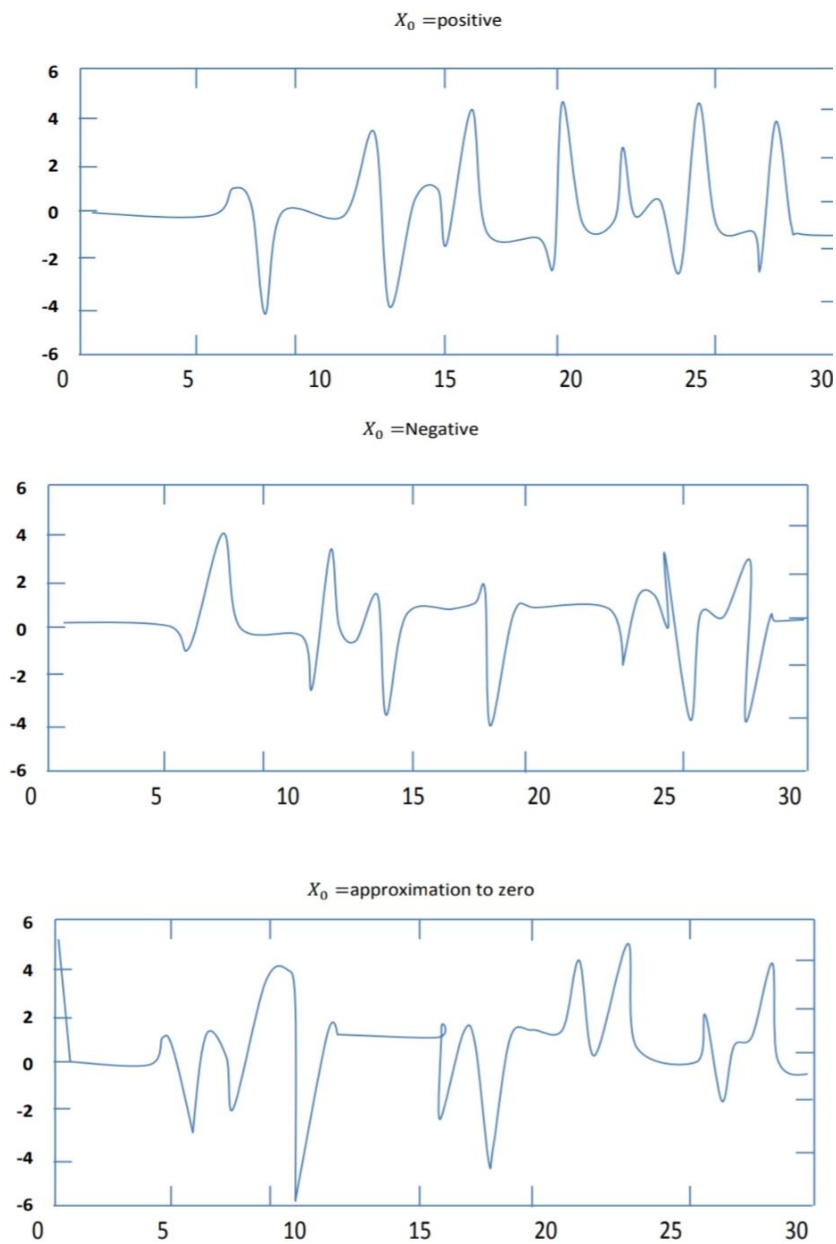

Figure 2

From this figure, the model does not have a limit cycle.

References

Abu Saleh, Mohammed Sobhi, (2000). *Introduction to statistical methods"*, Dar Al – yazouri scientific for publishing and distribution, Amman, Jordan.

Al-Badrani, Zafr Ramadan Mutar, (2002). Study in the diagnosis of coincidental control systems with special reference to the mode of state and stability space, *unpublished PhD thesis, College of Computer Science and Mathematics, University of Mosul.*

AL-Mahdaoui, Haifa Jafar, (1991). Unstable self-decline with higher degrees, *PhD thesis, College of management and economics, Mustansiriyah University.*

Chatfield, C. (1984). *The Analysis of Time Series an Introduction,* 3rd Ed., J.W. Arrow smith Ltd., Bristol, GB.

Hamilton, J.D. (1994). *Time Series Analysis,* Princeton university press, USA.

Ozaki, T. (1985). Nonlinear Time Series Models and Dynamical System, E.J. Hannan, P.R. Krishnaiah, M, M.Rao, eds., *Handbook of statistics, 5,* 25- 83.

Priestley, M.B. (1981). *Spectral Analysis and Time Series,* "Univariateseries", Academic Press. Inc. London, 1.

Ozaki, T. (1982). *Nonlinear Time-series Stochastic Processes and Dynamical system*. (Handbook of Statistics) 5 ltd.

Priestley, M.B. (1981). *Spectral Analysis and Time Series volume I Univariate Series* (Academic Press. Inc.) London.

Tong, H. (1990). *Nonlinear Time series; A Dynamical system Approach* (Oxford University Press) London.

Wen, S. (2015). *Introduction to Ordinary and Partial Differential Equations* (Spring).

William, F.T. (2013). *Elementary Differential Equations Wiyh Boundary Value Problem* (New-York).

Kawther, A.N. (2019). *Numerical Analysis & Methods Using Matlab,* IRAQ.

Alhayani, B. S., & Llhan, H. (2021). Visual sensor intelligent module based image transmission in industrial manufacturing for monitoring and manipulation problems. *Journal of Intelligent Manufacturing, 32,* 597-610.

Alhayani, B., & Ilhan, H. (2017). Hyper spectral image classification using dimensionality reduction techniques. *International Journal of Innovative Research in Electrical, Electronics, Instrumentation and Control Engineering, 5*(4), 71-74.

Al-Hayani, B., & Ilhan, H. (2020). Efficient cooperative image transmission in one-way multi-hop sensor network. *The International Journal of Electrical Engineering & Education, 57*(4), 321-339.

Al Hayani, B., & Ilhan, H. (2020). Image transmission over decode and forward based cooperative wireless multimedia sensor networks for Rayleigh fading channels in medical internet of things (MIoT) for remote health-care and health communication monitoring. *Journal of Medical Imaging and Health Informatics, 10*(1), 160-168.

Alhayani, B., Mohammed, H.J., Chaloob, I.Z., & Ahmed, J.S. (2021). Effectiveness of artificial intelligence techniques against cyber security risks apply of IT industry. *Materials Today: Proceedings.*

Alhayani, B., Abbas, S.T., Khutar, D.Z., & Mohammed, H.J. (2021). Best ways computation intelligent of face cyber attacks. *Materials Today: Proceedings.*

Alhayani, B., & Abdallah, A.A. (2020). Manufacturing intelligent Corvus corone module for a secured two way image transmission under WSN. *Engineering Computations.* https:/doi.org/10.1 10OREC12-2020-0107.

Alhayani, B., Abbas, S. T., Mohammed, H. J., & Mahajan, H. B. (2021). Intelligent secured two-way image transmission using corvus corone module over WSN. *Wireless Personal Communications,* 1-36. https://doi.org/10.1007s1277-021-08484-2.\