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Partial Metric Space with Modulation of Fixed Point Theorems in Generalized Contractions Mapping

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Abstract

In 2014, Jleli and Samet [6] introduced a contraction mapping called θ-contraction. After this, many authors had given their contribution on $\phi - \theta$-contraction (see [7, 8]). Motivated by [[6, 9]], we introduce the notion of generalized $\phi - \theta$ contraction and establish some new fixed point theorems for this contraction in the setting of complete partial metric spaces.

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1 Introduction

In 2014, Jleli and Samet [6] introduced a contraction mapping called θ-contraction. After this, many authors had given their contribution on $\phi-\theta$-contraction (see [7, 8]).

Inspired by [[6, 9]], we introduce the notion of generalized $\phi-\theta$ contraction and establish some new fixed point theorems for this contraction in the setting of complete partial metric spaces.

2 Preliminary Notes

A modulation of standard metric spaces is given in a partial metric space. By adding additional modifications, in 1992, Matthews [2, 5] invented the notion of a partial metric space in which $d(x,x)$ are not necessarily zero. Dhanorkar and Salunke [3] have proved fixed point theorem on partial metric space using continuous and monotonically non-decreasing mapping $\phi, \psi : [0,\infty) \to [0,\infty)$ with $\phi(0) = \psi(0) = 0$. Many and many researchers [4, 7, 8, 9].

In 2011 M.R.Ahmadi Zand and A.Dehghan Nezhad [1] introduce generalization of partial metric space. Dhanorkar and Salunke [4] proved a fixed point theorem in generalized partial metric spaces with nondecreasing map $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) < t$, for each $t > 0$.

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Definition 2.1 *[1] Let X be a nonempty set. Suppose a mapping g_{dp} : $X \times X \times X \to [0, +\infty)$ satisfies*

(1) $0 \leq g_{dp}(x, x, x) \leq g_{dp}(x, x, y) \leq g_{dp}(x, y, z)$ *for all* $x, y, z \in X$;
(2) $g_{dp}(x, y, z) = g_{dp}(z, x, y) = g_{dp}(x, z, y)$ *(Symmetric in all three variables)*;
(3) $g_{dp}(x, y, z) \leq g_{dp}(x, a, a) + g_{dp}(a, y, z) - g_{dp}(a, a, a)$ *for all* $x, y, z, a \in X$;
(4) $x = y = z$ *if* $g_{dp}(x, y, z) = g_{dp}(x, x, x) = g_{dp}(y, y, y) = g_{dp}(z, z, z)$ *for all* $x, y, z \in X$.

Then (X, g_{dp}) is called generalized partial metric space i.e. GP-metric space.

Example 2.2 *Let $X = [0, +\infty)$, and let $g_{dp} : X \times X \times X \to [0, +\infty)$, by $g_{dp}(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Clearly (X, g_{dp}) is not a G-metric space.*

Example 2.3 *Let $X = \{a, b, c\}$, and let $g_{dp} : X \times X \times X \to [0, +\infty)$, defined by $g_{dp}(x, y, z) = 1$,if $x = y = z$;*

$$g_{dp}(a, b, b) = g_{dp}(b, a, a) = 10;$$
$$g_{dp}(a, c, c) = g_{dp}(c, a, a) = 15;$$
$$g_{dp}(b, c, c) = g_{dp}(c, b, b) = 17;$$
$$g_{dp}(a, b, c) = 20.$$

Clearly (X, g_{dp}) is not a G-metric space.

Proposition 2.4 *[1] Let (X, g_{dp}) be a GP-metric space and for $x, y, z,$ and $a \in X$ then the following relations are true:*

(i) $g_{dp}(x, y, z) \leq g_{dp}(x, x, y) + g_{dp}(x, x, z) - g_{dp}(x, x, x)$;
(ii) $g_{dp}(x, y, y) \leq 2g_{dp}(x, x, y) - g_{dp}(x, x, x)$;
(iii) $g_{dp}(x, y, z) \leq g_{dp}(x, a, z) + g_{dp}(z, a, a) - g_{dp}(a, a, a)$;
(iv) $g_{dp}(x, y, z) \leq g_{dp}(a, a, x) + g_{dp}(a, a, y) + g_{dp}(a, a, z) - 2g_{dp}(a, a, a)$;
(v) $g_{dp}(x, y, z) + g_{dp}(a, a, a) \leq (2/3)(g_{dp}(x, y, a) + g_{dp}(x, a, z) + g_{dp}(a, y, z))$.

Proposition 2.5 *[1] Every GP-metric space (X, g_{dp}) defines a metric space $(X, d_{g_{dp}})$ as follows:*

$$d_{g_{dp}}(x, y) = g_{dp}(x, y, y) + g_{dp}(y, x, x) - g_{dp}(x, x, x) - g_{dp}(y, y, y),$$

for all $x, y \in X$.

Definition 2.6 *(i) A point $x \in X$ in GP-metric space (X, g_{dp}) is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$, if $\lim_{n,m\to\infty} g_{dp}(x, x_n, x_m) = g_{dp}(x, x, x)$. In this case, we say that the sequence $\{x_n\}$ is GP-convergent to x.*

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(ii)A sequence $\{x_n\}$ *in GP-metric space* (X, g_{dp}) *is said to be Cauchy iff* $\lim_{n,m,l\to\infty} g_{dp}(x_n, x_m, x_l)$ *is finite.*
(iii)A GP-metric space (X, g_{dp}) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$ *such that* $\lim_{n,m,l\to\infty} g_{dp}(x_n, x_m, x_l) = g_{dp}(x, x, x)$.

Proposition 2.7 *Let* (X, g_{dp}) *GP-metric space. Then for sequence then following are equivalent :*

$$(i)\ \{x_n\}\ is\ GP-convergent\ to\ x,$$
$$(ii)\ g_{dp}(x_n, x_n, x) \to g_{dp}(x, x, x)\ as\ n \to \infty,$$
$$(iii)\ g_{dp}(x_n, x, x) \to g_{dp}(x, x, x)\ as\ n \to \infty.$$

for all $x, y \in X$.

Main results

According to [[6, 9]], denote by Θ the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

$$(\Theta_1)\ \theta\ is\ nondecreasing,$$
$$(\Theta_2)\ For\ each\ sequence\ \{t_n\} \subset (0, \infty), \lim_{n\to\infty} \theta(t_1) = 1$$
$$if\ and\ only\ if\ \lim_{n\to\infty}(t_1) = 0^+,$$
$$(\Theta_3)\ \theta\ is\ continuous\ on\ (0, \infty).$$

And Φ the set of functions $\phi : [1, \infty) \to [1, \infty)$ satisfying the following conditions

$$(\Phi_1)\ \phi\ is\ nondecreasing,$$
$$(\Phi_2)\ For\ each\ t > 1, \lim_{n\to\infty}(\phi^n) = 1,$$
$$(\Phi_3)\ \theta\ is\ continuous\ on\ [1, \infty).$$

Lemma 2.8 *[9]. If* $\phi \in \Phi$, *then* $\phi(1) = 1$ *and* $\phi(t) < t$ *for each* $t > 1$.

Definition 2.9 *Let* (X, g_{dp}) *be a partial metric space. A mapping* $T : X \to X$ *is said to be a generalized* $\theta - \phi$ *contraction if there exist* $\theta \in \Theta$ *and* $\phi \in \Phi$ *such that, for any* $x, y, z \in X$

$$g_{dp}(Tx, Ty, Tz) \neq 0 \Rightarrow \theta(g_{dp}(Tx, Ty, Tz)) \leq \phi(\theta(M(x, y, z))) \tag{2.1}$$

where

$$M(x, y, z) = \max\Big\{g_{dp}(x, y, z), g_{dp}(x, Tx, Tx), g_{dp}(y, Ty, Ty), g_{dp}(z, Tz, Tz), \frac{1}{2}g_{dp}(x, Ty, Ty), \frac{1}{2}g_{dp}(y, Tz, Tz), \frac{1}{2}g_{dp}(z, Tx, Tx), \frac{1}{3}[g_{dp}(x, Ty, Ty) + g_{dp}(y, Tz, Tz) + g_{dp}(z, Tx, Tx)]\Big\} \tag{2.2}$$

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Theorem 2.10 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ is generalized $\theta - \phi$ contraction. Then T has a unique fixed point x^*, such that the sequence $\{T_n(x)\}$ converges to x^* for every $x \in X$.*

Let $x_0 \in X$ be an arbitrary point and define the sequence x_n in X by $Tx_n = x_{n+1}, \forall n \in N$. If $x_n = x_{n+1}$ for some $n \in N$, then $x_n = x^*$ is a fixed point for T. Now if , we $x_n \neq x_{n+1}$ for all $n \in N$.Then clearly $g_{dp}(Tx_n, Tx_{n+1}, Tx_{n+}) > 0$ for all n. From 2.1, we get

$$\theta(g_{dp}(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \phi(\theta(M(x_n, x_{n+1}, x_{n+1}))) \tag{2.3}$$

where

$$\begin{aligned}
&M(x_n, x_{n+1}, x_{n+1}) = \max\Big\{g_{dp}(x_n, x_{n+1}, x_{n+1}), g_{dp}(x_n, x_{n+1}, x_{n+1}), g_{dp}(x_{n+1}, x_{n+2}, x_{n+2}),\\
&g_{dp}(x_{n+1}, x_{n+2}, x_{n+2}), \frac{1}{2}g_{dp}(x_n, x_{n+2}, x_{n+2}), \frac{1}{2}g_{dp}(x_{n+1}, x_{n+2}, x_{n+2}), \frac{1}{2}g_{dp}(x_{n+1}, x_{n+1}, x_{n+1}),\\
&\frac{1}{3}[g_{dp}(x_n, x_{n+1}, x_{n+1}) + g_{dp}(x_{n+1}, x_{n+2}, x_{n+2}) + g_{dp}(x_{n+1}, x_{n+1}, x_{n+1})]\Big\}\\
&= \max\Big\{g_{dp}(x_n, x_{n+1}, x_{n+1}), g_{dp}(x_{n+1}, x_{n+2}, x_{n+2}), \frac{1}{2}g_{dp}(x_n, x_{n+2}, x_{n+2}), \frac{1}{2}g_{dp}(x_{n+1}, x_{n+1}, x_{n+1}),\\
&\frac{1}{3}[g_{dp}(x_n, x_{n+1}, x_{n+1}) + g_{dp}(x_{n+1}, x_{n+2}, x_{n+2}) + g_{dp}(x_{n+1}, x_{n+1}, x_{n+1})]\Big\}\\
&= \max\Big\{g_{dp}(x_n, x_{n+1}, x_{n+1}), g_{dp}(x_{n+1}, x_{n+2}, x_{n+2})\Big\}
\end{aligned} \tag{2.4}$$

If $M(x, y, z) = g_{dp}(x_{n+1}, x_{n+2}, x_{n+2})$, then using condition 2.1, we get

$$\begin{aligned}
(g_{dp}(x_{n+1}, x_{n+2}, x_{n+2})) &= \theta(g_{dp}(Tx_n, Tx_{n+1}, Tx_{n+1}))\\
&\leq \phi(\theta(M(x_n, x_{n+1}, x_{n+1})))\\
&= \phi(\theta(g_{dp}(x_{n+1}, x_{n+2}, x_{n+2})))\\
&< \theta(g_{dp}(x_{n+1}, x_{n+2}, x_{n+2})) \qquad (\because 2.8)
\end{aligned} \tag{2.5}$$

which is contradiction. Hence we can write for all $n \in N$, If $M(x, y, z) = g_{dp}(x_n, x_{n+1}, x_{n+1})$, then using condition 2.1, we get $\theta(g_{dp}(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \phi(\theta(g_{dp}(x_n, x_{n+1}, x_{n+1})))$ using this we get iterative sequence as

$$\begin{aligned}
(g_{dp}(x_n, x_{n+1}, x_{n+1})) &= \theta(g_{dp}(Tx_{n-1}, Tx_n, Tx_n))\\
&\leq \phi(\theta(g_{dp}(x_{n-1}, x_n, x_n)))\\
&\leq \phi^2(\theta(g_{dp}(x_{n-2}, x_{n-1}, x_{n-1})))\\
&\leq \phi^3(\theta(g_{dp}(x_{n-3}, x_{n-2}, x_{n-2})))\\
&\vdots\\
&\leq \phi^n(\theta(g_{dp}(x_0, x_1, x_1)))
\end{aligned} \tag{2.6}$$

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By the condition of Θ and Φ, we get

$$\lim_{n\to\infty} \phi^n(\theta(g_{dp}(x_0, x_1, x_1)) = 1 \tag{2.7}$$

$$and \lim_{n\to\infty} (g_{dp}(x_n, x_{n+1}, x_{n+1})) = 0 \tag{2.8}$$

Hence $\{x_n\}$ cauchy in complete metric space (X, g_{dp}). There exist x^* in X such that $x_n \to x^*$ such that $\lim_{n\to\infty} g_{dp}(x_n, x^*, x^*) = 0$. we can write

$$\lim_{n\to\infty} g_{dp}(x_{n+1}, Tx^*, Tx^*) = g_{dp}(x^*, Tx^*, Tx^*) \tag{2.9}$$

If $g_{dp}(x^{n+1}, Tx^*, Tx^*) \neq 0$, then by 2.1, we get

$$\theta(g_{dp}(x^{n+1}, Tx^*, Tx^*)) \leq \phi(\theta(M(x_n, x^*, x^*))) \tag{2.10}$$

where

$$\begin{aligned} M(x_n, x^*, x^*) = \max\Big\{ & g_{dp}(x_n, x^*, x^*), g_{dp}(x_n, x_{n+1}, x_{n+1}), g_{dp}(x^*, Tx^*, Tx^*), g_{dp}(x^*, Tx^*, Tx^*), \\ & \frac{1}{2} g_{dp}(x_n, Tx^*, Tx^*), \frac{1}{2} g_{dp}(x^*, Tx^*, Tx^*), \frac{1}{2} g_{dp}(x^*, x_{n+1}, x_{n+1}), \\ & \frac{1}{3}[g_{dp}(x_n, Tx^*, Tx^*) + g_{dp}(x^*, Tx^*, Tx^*) + g_{dp}(x^*, x_{n+1}, x_{n+1})]\Big\} \\ = {} & g_{dp}(x^*, Tx^*, Tx^*) \qquad (as\ n \to \infty) \end{aligned} \tag{2.11}$$

above condition 2.10 becames

$$\begin{aligned} \theta(g_{dp}(x^{n+1}, Tx^*, Tx^*)) &\leq \phi(\theta(g_{dp}(x^*, Tx^*, Tx^*))) \\ &< g_{dp}(x^*, Tx^*, Tx^*) \end{aligned}$$

which is a contradiction, hence $Tx^* = x^*$.
Uniqueness: If suppose there exist y^* such that $y^* \neq x^*$ and $Ty^* = y^*$. Therefore $g_{dp}(Tx^*, Ty^*, Ty^*) = g_{dp}(x^*, y^*, y^*) > 0$ and $M(x^*, y^*, y^*)) = g_{dp}(x^*, y^*, y^*)$, we get

$$\begin{aligned} \theta(g_{dp}(x^*, y^*, y^*)) &= \theta(g_{dp}(Tx^*, Ty^*, Ty^*)) \\ &\leq \phi(\theta(M(x^*, y^*, y^*))) \\ &= \leq \phi(\theta(g_{dp}(x^*, y^*, y^*))) \\ &< g_{dp}(x^*, y^*, y^*) \end{aligned}$$

which is a contradiction. Hence, T having unique fixed point.

Theorem 2.11 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ is generalized $\theta - \phi$ contraction such that for $x, y \in X$*

$$g_{dp}(Tx, Ty, Ty) \neq 0 \Rightarrow \theta(g_{dp}(Tx, Ty, Ty)) \leq \phi(\theta(M(x, y, y))) \tag{2.12}$$

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where

$$M(x,y,y) = \max\Big\{g_{dp}(x,y,y), g_{dp}(x,Tx,Tx), g_{dp}(y,Ty,Ty), \frac{1}{2}g_{dp}(x,Ty,Ty), \frac{1}{2}g_{dp}(y,Tx,Tx), \frac{1}{3}[g_{dp}(x,Ty,Ty) + g_{dp}(y,Ty,Ty) + g_{dp}(y,Tx,Tx)]\Big\} \quad (2.13)$$

Then T has a unique fixed point x^*, such that the sequence $\{T_n(x)\}$ converges to x^* for every $x \in X$.

One can prove easily using 2.10.

Theorem 2.12 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ satisfying following condition such that for all $x, y \in X$*

$$g_{dp}(Tx,Ty,Ty) \le \max\Big\{a g_{dp}(x,y,y), b\Big(g_{dp}(x,Tx,Tx) + 2g_{dp}(y,Ty,Ty)\Big), b\Big(g_{dp}(x,Ty,Ty) + g_{dp}(y,Ty,Ty) + g_{dp}(y,Tx,Tx)\Big)\Big\} \quad (2.14)$$

where $0 \le a < 1$ and $0 \le b < \frac{1}{3}$. Then T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

To prove this result we will use $\theta-\phi$ contraction by defining $\theta(t) = e^t$, $\phi(t) = t^\lambda$, where $\lambda = \max a, 3b$, then $0 \le \lambda < 1$. Clearly $\theta \in \Theta$ and $\phi \in \Phi$. Since

$$\begin{aligned}
&\max\Big\{a g_{dp}(x,y,y), b\Big(g_{dp}(x,Tx,Tx) + 2g_{dp}(y,Ty,Ty)\Big), \\
&\qquad b\Big(g_{dp}(x,Ty,Ty) + g_{dp}(y,Ty,Ty) + g_{dp}(y,Tx,Tx)\Big)\Big\} \\
&\le \lambda \max\Big\{g_{dp}(x,y,y), \frac{1}{3}\Big(g_{dp}(x,Tx,Tx) + 2g_{dp}(y,Ty,Ty)\Big), \\
&\qquad \frac{1}{3}\Big(g_{dp}(x,Ty,Ty) + g_{dp}(y,Ty,Ty) + g_{dp}(y,Tx,Tx)\Big)\Big\} \\
&\le \lambda \max\Big\{g_{dp}(x,y,y), g_{dp}(x,Tx,Tx) + g_{dp}(y,Ty,Ty), \\
&\qquad \frac{1}{3}\Big(g_{dp}(x,Ty,Ty) + g_{dp}(y,Ty,Ty) + g_{dp}(y,Tx,Tx)\Big)\Big\} \\
&\le \lambda M(x,y,y). \qquad (2.15)
\end{aligned}$$

Hence

$$\begin{aligned}
\theta(g_{dp}(Tx,Ty,Ty)) &= e^{g_{dp}(Tx,Ty,Ty)} \le e^{\lambda M(x,y,y)} \\
&= (e^{M(x,y,y)})^\lambda = \phi(\theta(M(x,y,y))) \qquad (2.16)
\end{aligned}$$

using theorem 2.10, we get T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

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Theorem 2.13 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ satisfying following condition such that for all $x, y \in X$*

$$g_{dp}(Tx, Ty, Ty) \leq k \max\Big\{g_{dp}(x, y, y), g_{dp}(x, Tx, Tx), g_{dp}(y, Ty, Ty), g_{dp}(z, Tz, Tz), g_{dp}(x, Ty, Ty), g_{dp}(y, Tz, Tz), g_{dp}(z, Tx, Tx)\Big\} \tag{2.17}$$

where $0 \leq k < \frac{1}{2}$. Then T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

To prove this result, we will use $\theta - \phi$ contraction by defining $\theta(t) = e^t$, $\phi(t) = t^\lambda$, where $\lambda = 2k$, then $0 \leq \lambda < 1$. Clearly $\theta \in \Theta$ and $\phi \in \Phi$. Since

$$\begin{aligned} k \max\Big\{&g_{dp}(x, y, y), g_{dp}(x, Tx, Tx), g_{dp}(y, Ty, Ty), g_{dp}(z, Tz, Tz), \\ &g_{dp}(x, Ty, Ty), g_{dp}(y, Tz, Tz), g_{dp}(z, Tx, Tx)\Big\} \\ &\leq \lambda\Big\{\frac{1}{2}g_{dp}(x, y, y), \frac{1}{2}g_{dp}(x, Tx, Tx), \frac{1}{2}g_{dp}(y, Ty, Ty), \frac{1}{2}g_{dp}(z, Tz, Tz), \\ &\frac{1}{2}g_{dp}(x, Ty, Ty), \frac{1}{2}g_{dp}(y, Tz, Tz), \frac{1}{2}g_{dp}(z, Tx, Tx)\Big\} \\ &\leq \lambda M(x, y, z). \end{aligned} \tag{2.18}$$

Hence

$$\begin{aligned} \theta(g_{dp}(Tx, Ty, Tz)) &= e^{g_{dp}(Tx,Ty,Tz)} \leq e^{\lambda M(x,y,z)} \\ &= (e^{M(x,y,z)})^\lambda = \phi(\theta(M(x, y, z))) \end{aligned} \tag{2.19}$$

using theorem 2.10, we get T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

Now by using definition 2.9, we can easily prove following theorem.

Theorem 2.14 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ be a self mapping, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y, z \in X$*

$$g_{dp}(Tx, Ty, Tz) \neq 0 \Rightarrow \theta(g_{dp}(Tx, Ty, Tz) \leq \phi(\theta(g_{dp}(x, y, z) \tag{2.20}$$

Then T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

Using Theorem 2.14, we can easily prove theorem stated by Mustafa, Khandagji and Shatanawi [?] as

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Theorem 2.15 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ be a self mapping, there exist $\lambda \in [0, 1)$ such that*

$$g_{dp}(Tx, Ty, Tz) \leq \lambda g_{dp}(x, y, z) \tag{2.21}$$

for all $x, y, z \in X$. Then T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

Proof: Let $\theta(t) = e^t$ and $\phi(t) = t^\lambda$, then $\theta \in \Theta$ and $\phi \in \Phi$. Since

$$g_{dp}(Tx, Ty, Tz) \leq \lambda g_{dp}(x, y, z)$$

gives $e^{g_{dp}(Tx,Ty,Tz)} \leq e^{\lambda g_{dp}(Tx,Ty,Tz)} = (e^{g_{dp}(Tx,Ty,Tz)})^\lambda$, follows $\theta(g_{dp}(Tx, Ty, Tz) \leq \phi(\theta(g_{dp}(x, y, z)$.
using theorem 2.15, we get T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

Corollary 2.16 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ be a self mapping, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y, z \in X$*

$$g_{dp}(Tx, Ty, Ty) \neq 0 \Rightarrow \theta(g_{dp}(Tx, Ty, Ty)) \leq \phi(\theta(g_{dp}(x, y, y)) \tag{2.22}$$

Then T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

Corollary 2.17 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ be a self mapping, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y, z \in X$*

$$g_{dp}(Tx, Ty, Tz) \neq 0 \Rightarrow \theta(g_{dp}(Tx, Ty, Tz) \leq \phi\Big(\theta\Big(\frac{g_{dp}(x, Tx, Tx) + g_{dp}(y, Ty, Ty) + g_{dp}(z, Tz, Tz)}{3}\Big)\Big) \tag{2.23}$$

Then T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

Corollary 2.18 *Let (X, g_{dp}) be a complete partial metric space and let $T : X \to X$ be a self mapping, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$*

$$g_{dp}(Tx, Ty, Ty) \neq 0 \Rightarrow \theta(g_{dp}(Tx, Ty, Ty) \leq \phi\Big(\theta\Big(\frac{g_{dp}(x, Tx, Tx) + g_{dp}(y, Ty, Ty)}{2}\Big)\Big) \tag{2.24}$$

Then T has a unique fixed point x^* such that the sequence $\{T^n(x)\}$ converges to x^* for every $x \in X$.

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Example 2.19 *Let* $X = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$ *and metric* $g_{dp}(x, y, z) = \max\{|x|, |y|, |z|\}$ *for all* $x, y, z \in X$. *Then* (X, g_{dp}) *is a complete* g_{dp}*-metric space. Define the mapping* $T : X \longrightarrow X$ *by* $Tx = \begin{cases} 0, & x=0; \\ -(n-1), & x=n; \\ (n-1), & x=-n. \end{cases}$
Now, let the function $\theta : (0, \infty) \longrightarrow (1, \infty)$ *defined by* $\theta(t) = 5^t$ *and the function* $\phi : [1, \infty) \longrightarrow [1, \infty)$ *defined by* $\phi(t) = \begin{cases} 3, & 1 \le t \le 2; \\ t^2 - 1, & t \ge 2. \end{cases}$
Clearly $\theta \in \Theta$ *and* $\phi \in \Phi$.
We consider following cases
Case 1: *If* $(x = n \ge 1, y = 0$ *or* $x = -n(n \ge 1), y = 0)$.

$$\begin{aligned} &g_{dp}(Tx, Ty, Ty) = n - 1, \\ &g_{dp}(x, Tx, Tx) = n, \\ &g_{dp}(y, Ty, Ty) = 0, \\ &\theta(g_{dp}(Tx, Ty, Ty)) = \theta(n-1) = 5^{n-1} \end{aligned}$$

$$\begin{aligned} &\phi\Big(\theta\Big(\frac{g_{dp}(x, Tx, Tx) + g_{dp}(y, Ty, Ty)}{2}\Big)\Big) = \phi\Big(\theta\Big(\frac{n}{2}\Big)\Big) \\ &= \phi\Big(5^{\frac{n}{2}}\Big) = 5^n - 1 \\ &\ge 5^{n-1} = \theta(g_{dp}(Tx, Ty, Ty)) \end{aligned} \tag{2.25}$$

Case 2: $(x = n > y = m \ge 1$ *or* $x = -n < y = -m \le -1)$.

$$\begin{aligned} &g_{dp}(Tx, Ty, Ty) = n - 1, \\ &g_{dp}(x, Tx, Tx) = n, \\ &g_{dp}(y, Ty, Ty) = m, \\ &\theta(g_{dp}(Tx, Ty, Ty)) = \theta(n-1) = 5^{n-1} \end{aligned}$$

$$\begin{aligned} &\phi\Big(\theta\Big(\frac{g_{dp}(x, Tx, Tx) + g_{dp}(y, Ty, Ty)}{2}\Big)\Big) = \phi\Big(\theta\Big(\frac{n+m}{2}\Big)\Big) \\ &= \phi\Big(5^{(\frac{n+m}{2})}\Big) = 5^{n+m} - 1 \\ &\ge 5^{n-1} = \theta(g_{dp}(Tx, Ty, Ty)) \end{aligned} \tag{2.26}$$

Case 3: $(x = n, y = -m, n > m \ge 1$ *or* $x = -n, y = m, n \ge m \ge 1)$.

$$\begin{aligned} &g_{dp}(Tx, Ty, Ty) = n - 1, \\ &g_{dp}(x, Tx, Tx) = n, \\ &g_{dp}(y, Ty, Ty) = m, \\ &\theta(g_{dp}(Tx, Ty, Ty)) = \theta(n-1) = 5^{n-1} \end{aligned}$$

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$$\phi\Big(\theta\Big(\frac{g_{dp}(x,Tx,Tx)+g_{dp}(y,Ty,Ty)}{2}\Big)\Big)=\phi\Big(\theta\Big(\frac{n+m}{2}\Big)\Big)$$
$$=\phi\Big(5^{(\frac{n+m}{2})}\Big)=5^{n+m}-1$$
$$\geq 5^{n-1}=\theta(g_{dp}(Tx,Ty,Ty)) \qquad (2.27)$$

From 2.25, 2.26 *and* 2.27*, we get for all* $x, y \in X$*,*

$$\theta(g_{dp}(Tx,Ty,Ty))\leq\phi\Big(\theta\Big(\frac{g_{dp}(x,Tx,Tx)+g_{dp}(y,Ty,Ty)}{2}\Big)\Big)$$

This gives Corollary 2.18 satisfied, thus T *has a fixed point at* $x = 0$.

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